HEAT EXCHANGE IN PLANE LAMINAR FLOW OF A VISCOUS FLUID

(TEPLOOBMEN V PLOSKOM USTANOVIVSHEMSIA POTOKE VISKOI ZHIDKOSTI)

PMM Vol.23, No.3, 1959, pp. 581-582

V.I. MERKULOV
(Kiev)

(Received 9 October 1958)

In this article the equation of steady heat exchange in a laminar flow of a viscous fluid is considered. It is assumed that the heat conduction coefficient is constant and that the problem regarding the determination of the velocity field of the flows considered has been solved.

The stationary temperature distribution in a laminar flow is governed by the equation [1]

$$v_x \frac{\partial T}{\partial x} + v_y \frac{\partial T}{\partial y} = a\Delta T + f(x, y) \qquad (1)$$

where v_x and v_y are the components of the velocity vector. If a body with a boundary S is placed in a flow of infinite extent, then the boundary conditions for equation (1) will be

$$\alpha T + \beta \frac{\partial T}{\partial n} = \gamma \quad \text{on } S, \qquad \lim T = \text{const}, \quad x^2 + y^2 \to \infty$$

Obviously, the last constant may always be made zero. Further, in the following we shall consider only Dirichlet's problem, that is $\beta = 0$.

We introduce the flow function Ψ. Equation (1) may then be written in the form

$$\frac{D(T, \Psi)}{D(x, y)} = a\Delta T + f(x, y), \qquad v_x = \frac{\partial \Psi}{\partial y}, \qquad v_y = -\frac{\partial \Psi}{\partial x} \qquad (2)$$

Equation (2) remains of the same form during the passage to any other arbitrary isothermal system of coordinates. Let us pass, for example, to the system of coordinates ϕ, ψ, where ϕ and ψ are the real and imaginary parts of the complex ideal flow potential in the same region. In the new system we obtain the equation

$$\frac{D(T, \Psi)}{D(\varphi, \psi)} = a\Delta_{\varphi\psi} T + f_1(\varphi, \psi) \qquad (3)$$

and the region Q with boundary S will pass over into a plane Q' with a cut S' along the ϕ-axis, on which the boundary conditions should be satisfied.

We introduce the function $\chi = \psi - \Psi$ and rewrite equation (3) as

$$\frac{1}{a}\frac{\partial T}{\partial \varphi} - \triangle T = \frac{1}{a}\frac{D(T, \chi)}{D(\varphi, \psi)} \tag{4}$$

We consider now the following differential equation, which governs the problem of heat exchange in ideal fluid flow

$$\frac{1}{a}\frac{\partial T}{\partial \varphi} - \triangle T = 0 \tag{5}$$

Several authors [2-4] studied this problem.

The middle of the slit S' shall be placed at the origin of coordinates and we introduce a system of elliptic coordinates ξ, η in accordance with formulas $\phi = \operatorname{ch} \xi \cos \eta$, $\psi = \operatorname{sh} \xi \sin \eta$.

The slit T' coincides thereby with the coordinate line $\xi = 0$. This makes it possible to solve equation (5) by the method of variables separable. Having the general solution of equation (5) and a particular solution, given by the function

$$\exp\left[-\frac{1}{2a}(\varphi - \varphi_0)\right] K_0\left[\frac{1}{2a}\sqrt{(\varphi-\varphi_0)^2 + (\psi-\psi_0)^2}\right]$$

we can construct Green's function for equation (5) of Dirichlet's problem in the plane Q' with a slit S'. In the notation of [5], the required function may be written as

$$G(\xi, \eta, \xi_0, \eta_0) = \frac{1}{2\pi} K_0\left(\frac{r}{2a}\right) \exp\left[-\frac{1}{2a}(\varphi - \varphi_0)\right] - g(\xi, \eta)$$

where

$$g(\xi, \eta) = \exp\left[-\frac{1}{2a}(\varphi - \varphi_0)\right] \sum_{m=0}^{\infty} [\alpha_m \operatorname{se}(\eta) + \beta_m \operatorname{ce}_m(\eta)]\, FeK_m(\xi)$$

$$\alpha_m = \frac{1}{2\pi FeK_m(0)} \int_{-\pi}^{\pi} K_0\left(\frac{r}{2a}\right)\Big|_{\xi=0} \operatorname{se}_m(\eta)\, d\eta, \qquad r^2 = (\varphi - \varphi_0)^2 + (\psi - \psi_0)^2$$

$$\beta_m = \frac{1}{2\pi FeK_m(0)} \int_{-\pi}^{\pi} K_0\left(\frac{r}{2a}\right)\Big|_{\xi=0} \operatorname{ce}_m(\eta)\, d\eta$$

With the aid of this function, equation (5) may be replaced by the integral equation

$$T(\varphi_0, \psi_0) = F(\varphi_0\psi_0) + \frac{1}{a}\iint_{Q'} T(\varphi, \psi)\frac{D(\chi, G)}{D(\varphi, \psi)}\, d\varphi\, d\psi \tag{6}$$

where $F(\phi_0, \psi_0)$ is the temperature field in ideal fluid flow.

The functions $\partial\chi/\partial\phi$ and $\partial\chi/\partial\psi$, which are the differences of flow velocities of viscous and ideal fluids, may be determined for continuous flows with large Reynolds numbers using the boundary layer theory. In accordance with this theory the indicated quantities will be equal to zero outside a certain parabola Q_1, whose equation is

$$\sqrt{(\varphi+c)^2+\psi^2}-(\varphi+c)^2=b\,R^{-1/2},$$

where R is Reynolds number and the parameter c is chosen in such a manner, that the slit S' is located in Q_1.

Further, in the parabola indicated one can put

$$\frac{\partial\chi}{\partial\varphi}\sim R^{-1/2},\qquad \frac{\partial\chi}{\partial\psi}\sim 1 \tag{7}$$

Moreover, from the condition $G=0$ at $\psi=0$ it follows, that $G\sim R^{-1/2}$ in Q_1. Obviously, the derivative $\partial G/\partial\phi$ in Q_1 will also be of the same order. All of this makes it possible to indicate the order of the root of the integral equation (6)

$$\frac{\partial\chi}{\partial\varphi}\frac{\partial G}{\partial\psi}-\frac{\partial\chi}{\partial\psi}\frac{\partial G}{\partial\varphi}\sim R^{-1/2}$$

Equation (6) will be solved in the space of bounded continuous functions on the plane $C[-\infty,\infty]$. The norm of the root of equation (6) will be determined by

$$\|K\|_C=\max_{(\varphi_0,\psi_0)}\left\{\frac{1}{a}\iint\limits_{Q_1}\left|\frac{D(\chi,G)}{D(\varphi,\psi)}\right|d\varphi\,d\psi\right\} \tag{8}$$

The variables ϕ, ψ and ϕ_0, ψ_0 will be substituted by

$$\varphi_1=\frac{\varphi}{a},\qquad \psi_1=\frac{\psi}{a},\qquad \varphi_{01}=\frac{\varphi_0}{a},\qquad \psi_{01}=\frac{\psi_0}{a}$$

With the scale chosen the function G will not depend on the parameter a.

Obviously, the estimates (7) retain their sense also for a new scale.

Using the representation of Green's function written earlier, as well as the estimates indicated, it is possible to obtain the inequality

$$\|K\|_C<M\,R^{-1/2}$$

where M is some constant, which does not depend on a and R.

This inequality permits to establish the theorem: equation (6), which governs the problem of heat exchange in a flow of a viscous fluid for sufficiently large Reynolds numbers, possesses a unique solution in the space $C[-\infty,\infty]$. This solution may be found by the method of successive approximations.

The free term $F(\phi_0, \psi_0)$, giving the solution of the problem of heat exchange in a flow of an ideal fluid, may differ but little from the solution of the problem of the viscous fluid in the sense of the metric $C[-\infty, \infty]$, if the number R is sufficiently large.

From these results it becomes clear, why the attempt of King [3] to solve the problem of the thermoanemometer using an ideal fluid model, was not successful.

BIBLIOGRAPHY

1. Targ, S.M., *Osnovnve zadachi teorii laminarnykh potokov (The Basic Problems of the Theory of Laminar Flows)*. GTTI, 1951.

2. Boussinesq, M.I., Calcul du pouvoir refroidissant des courants fluides. *J. Math.* Vol. 1, 1905.

3. King, L.V., On the convection of heat from small cylinders in a stream of fluid. *Phil. Trans.* A 214, 1914.

4. Sretenskii, L.N., O nagrevanii potoka zhidkosti tverdymi stenkami (On the heating of fluid flow by rigid walls). *PMM* Vol. 2, No. 2, 1935.

5. MacLachlan, *Theory and Application of Mathieu Functions*. Izd. inostr. lit., Moscow, 1935.

Translated by G.H.